Three lectures on elliptic surfaces and curves of high rank
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Over the past two years we have improved several of the (Mordell–Weil) rank records for elliptic curves over $\mathbf{Q}$ and nonconstant elliptic curves over $\mathbf{Q}(t)$. For example, we found the first example of a curve $E/\mathbf{Q}$ with 28 independent points $P_i \in E(\mathbf{Q})$ (the previous record was 24, by R. Martin and W. McMillen 2000), and the first example of a curve over $\mathbf{Q}$ with Mordell–Weil group $\cong (\mathbf{Z}/2\mathbf{Z}) \oplus \mathbf{Z}^{18}$ (the previous rank record for a curve with a 2-torsion point was 15, by Dujella 2002). In these lectures we give some of the background, theory, and computational tools that led to these new records and related applications.

I Context and overview: the theorems of Mordell(–Weil) and Mazur; the rank problem; the approaches of Néron–Shioda and Mestre; elliptic surfaces and Néron specialization; fields other than $\mathbf{Q}$.

II Elliptic surfaces and K3 surfaces: the Mordell–Weil and Néron–Severi groups; K3 surfaces of high Néron–Severi rank and their moduli; an elliptic K3 surface over $\mathbf{Q}$ of Mordell–Weil rank 17. Some other applications of K3 surfaces of high rank and their moduli.

III Computational issues, techniques, and results: slices of Niemeier lattices; finding and transforming models of K3 surfaces of high rank; searching for good specializations. Summary of new rank records for elliptic curves.

I Context and overview. Mordell (1922) proved that the set $E(\mathbf{Q})$ of rational points of an elliptic curve $E/\mathbf{Q}$ has the structure of an abelian group, and that this group is finitely generated. That is, $E(\mathbf{Q}) \cong T \oplus \mathbf{Z}^r$, where T is a finite abelian group (the *torsion group* of E) and r is the *rank* of E.[1] This raises the basic structural question:

Which groups arise as $E(\mathbf{Q})$ for some elliptic curve $E/\mathbf{Q}$?

Equivalently,

Which ordered pairs (T, r) arise as the torsion and rank of some elliptic curve $E/\mathbf{Q}$?

Mazur's celebrated torsion theorem [Mazur 1977] answers the questions of which torsion groups arise: the cyclic groups of order N for $1 \le N \le 10$ and $N = 12$, and the groups $(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/2N\mathbf{Z})$ for $1 \le N \le 4$. These are exactly the fifteen groups T for which there is a rational modular curve parametrizing elliptic curves E with an embedding of T into $E(\mathbf{Q})$. It is thus almost immediate that each of these fifteen groups arises infinitely often; the deep part of Mazur's theorem is the proof

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[1] Weil later (1928) generalized this from $E/\mathbf{Q}$ to A/K for an arbitrary abelian variety A over a number field K, which is why the group $E(\mathbf{Q})$ and rank r are often called the "Mordell–Weil group" and "Mordell–Weil rank" even in the case covered by Mordell's original result.

that when the modular curve has positive genus it has no rational points other than "cusps" (which parametrize certain degenerate elliptic curves).

This leaves the question of which values of r arise for each of the fifteen possible T. At present this question is hopelessly difficult. It is not even known whether infinitely many r arise; equivalently, whether $\limsup_{E/\mathbf{Q}}(r)$ is infinite for some (all) of those T. As long as this question remains intractable, we also ask for which (T, r_0) can we prove that $\limsup(r) \geq r_0$; this is in some sense a more demanding question than finding large individual values of r, in that proving $\limsup(r) \geq r_0$ requires infinitely many curves, not a single lucky guess.

Our main theme is the use of K3 surfaces of high rank and their moduli to get new records for these questions (and also to obtain some other applications of explicit parametrizations of K3 surfaces). For the remainder of this first lecture we outline how the quest for curves of large rank naturally leads to elliptic surfaces, and illustrate two important earlier approaches to the problem.

Essentially the only technique known for proving lower bounds on $\limsup(r)$ (at any rate the only technique known for $r_0 > 2$) is finding *parametrized families*, that is, infinite families of elliptic curves E together with generically independent points $P_1, \ldots, P_{r_0}$.

Paradigmatic example: given (x_i, y_i) $(i = 1, 2, 3)$, solve the simultaneous linear equations for a_2, a_4, a_6 that make $y_i^2 = x_i^3 + a_2 x_i^2 + a_4 x_i + a_6$ for each $i = 1, 2, 3$. This yields an elliptic curve E with 3 rational points (x_i, y_i). Exercise: they are generically independent (that is, independent when E is considered as an elliptic curve over the field $\mathbf{Q}(x_1, y_1, x_2, y_2, x_3, y_3)$). Hint: *any* quadruple (E, P_1, P_2, P_3) (E some elliptic curve, each P_i on E) arises this way for some x_i, y_i if and only if each $P_i \neq 0$ and $P_i \neq \pm P_j$ for $i \neq j$. [Moreover, we can make x_i, y_i unique by requiring $(x_1, x_2) = (0, 1)$; then (x_3, y_1, y_2, y_3) gives a birational parametrization of the "3-rd power $\mathcal{E}^3$ of the universal elliptic curve".] By a specialization theorem of Néron ([Néron 1952], see also [Serre 1989, Ch.11]), later sharpened by Silverman (more on this below), there are infinitely many choices of $(x_i, y_i) \in \mathbf{Q}^6$ for which P_1, P_2, P_3 remain independent on the curve $E/\mathbf{Q}$. Indeed (and not surprisingly), this is true for "most" rational (x_i, y_i), and infinitely many non-isomorphic curves E arise this way. Hence $\limsup(r) \geq 3$.

One quickly sees ways to improve this beyond rank 3; for instance, use the extended Weierstrass form $y^2 + a_1 x y + a_3 = x^3 + a_2 x^2 + a_4 x + a_6$ (or even the same with $a_0 x^3$ instead of x^3); or, pass a cubic plane curve through 9 "random" points of $\mathbf{P}^2$. These are still not that far from each other, but they do suggest two complementary ways of viewing the task. In general, an elliptic curve over $F(t_1, ..., t_n)$ with several rational points is both:

1) ["à la Mestre"] Polynomial identities (algebra, often ingeniously applied), and

2) ["à la Néron"] An algebraic variety of dimension $n + 1$ equipped with a suitable map to n-dimensional space over F (algebraic geometry).

We next give a short table of record ranks of nonconstant elliptic curves[2] over $\mathbf{Q}(t)$, all but the first and last of whose rows are taken from [Rubin–Silverberg 2002, Table 3] and represent the Mestre-style algebraic approach:

Rank $\geq$	Author(s) and year
8, 9, 10	Néron (1952)
11, 12	Mestre (1991)
13	Nagao (1994)
14	Mestre, Kihara (2000–1)
(15,16,)17,18	NDE (2006–7)

This leap from 14 to 18,[3] and similar improvements for curves with nontrivial torsion, is also the key ingredient (via specialization) of the new record ranks for individual curves over $\mathbf{Q}$. Curiously these improvements are achieved by returning to Néron's geometric viewpoint but applying it to elliptic surfaces at the next level of complexity: elliptic K3 surfaces rather than rational elliptic surfaces. We shall say more about this in the second lecture. First we interpolate some comments about Néron's family, an example of Mestre's identities, and remarks on elliptic curves and surfaces defined over number fields other than $\mathbf{Q}$.

Recall that we obtained rank ≥ 3 by birationally parametrizing *all* elliptic curves E with three rational points P_1, P_2, P_3, a.k.a. the "3-rd power $\mathcal{E}^3$ of the universal elliptic curve"; and observed that some higher powers can be likewise parametrized using other models of elliptic curves, notably the unique cubic curve passing through 9 general points $P_1, \ldots, P_9$ in the plane. This already gives $\limsup(r) \geq 9$, because there are various ways to get a 10th point P_0 that can serve as an origin, such as the 9th base-point of the pencil of cubics through $P_1, ..., P_8$.

For $r = 10$, and thus for larger r, it is no longer possible to completely parametrize $\mathcal{E}^r$ — ultimately because the modular form $\Delta = q \prod_{n=1}^{\infty}(1 - q^n)^{24}$ yields a holomorphic 11-form on $\mathcal{E}^{10}$! Nevertheless, Néron used the geometry of cubic curves to find (in effect) rational curves on $\mathcal{E}^{10}$ that give rise to elliptic curves over $\mathbf{Q}(t)$ with 10 generically independent rational points. We describe this construction in some detail because some of the ideas will recur in the more complicated setting of elliptic K3 surfaces.

We follow the exposition in [Shioda 1991]. Start with $P_1, ..., P_8$, and thus also the ninth base-point P_0. Then blowing up $\mathbf{P}^2$ at $P_0, P_1, \ldots, P_8$ gives a birational isomorphism of $\mathbf{P}^2$ with the pencil of cubics through these nine points. Choose $P_0, ..., P_8$ on a *cuspidal* cubic, say $\Gamma : Y^2 Z = X^3$, and choose the coordinate t on the pencil so that Γ is the preimage of $t = \infty$. For generic such $P_1, \ldots, P_8$ this surface has no reducible fibers, and so rank 8. Parametrize Γ by $A(u) = (u : 1 : u^3)$, so that $A(u_1), A(u_2), A(u_3)$ are collinear iff $u_1 + u_2 + u_3 = 0$; in particular, the

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[2] An elliptic curve over $\mathbf{Q}(t_1, \ldots, t_n)$ is "nonconstant" if it is not isomorphic over $\mathbf{Q}(t_1, \ldots, t_n)$ with a curve over $\mathbf{Q}$. Such a curve of rank r yields nonconstant curves of rank at least r over $\mathbf{Q}(t_1, \ldots, t_m)$ for each positive $m < n$ by specialization, so it is enough to consider $n = 1$.

[3] Ranks 15 and 16 are in parentheses because we proved the existence of such curves in 2006 but did not compute them explicitly.

line through $A(u)$ and $A(-u/2)$ is tangent to Γ at $A(-u/2)$. Let D_1, D_2, D_3 be these tangents for u_1, u_2, u_3. Each D_i meets a generic curve E_t of the pencil at 2 points other than $P_i = A(u_i)$; so we get a "double section": a section defined over a double cover of the t-line. Moreover, each of these double covers is rational, and $t = \infty$ is a branch point. So any two of them give a degree-4 cover of $\mathbf{P}^1$ by a rational curve, and Néron shows that the two new points[4] are independent, giving rank 10 over $\mathbf{Q}(t)$. For each $t \in \mathbf{Q}$ (with finitely many exceptions where E degenerates), we obtain by specialization an elliptic curve $E_t/\mathbf{Q}$ with 10 rational points. Since $E/\mathbf{Q}(t)$ is nonconstant, Néron's specialization theorem yields infinitely many choices of t where these points remain independent and the curves E_t are pairwise non-isomorphic. (Silverman later used the canonical height to construct, given a curve $E/\mathbf{Q}(t)$ and independent rational points $P_1, \ldots, P_n$, an effective bound H such that the specialized points on E_t remain independent for all t not of the form t_0/t_1 with $t_0, t_1 \in \mathbf{Z} \cap [-H, H]$, proving that the set of exceptions to independence is finite and effectively computable. See again [Serre 1989, Ch.11].) Using all three D_i yields rank 11 over the compositum of three rational double covers of $\mathbf{Q}(t)$, all branched at $t = \infty$. This compositum is the function field of an elliptic curve, usually of positive rank and thus giving infinitely many examples of elliptic curves over $\mathbf{Q}$ with 11 rational points. A variation of Néron's specialization theorem, or of Silverman's refinement, then shows that these include infinitely many distinct curves of rank at least 11 over $\mathbf{Q}$.

But this did not quite give a nonconstant elliptic curve of rank ≥ 11 over $\mathbf{Q}(t)$. Such a curve was first constructed in [Mestre 1991], as follows. Suppose we have distinct $x_1, \ldots, x_{12} \in \mathbf{Q}$, polynomials $A_2, A_3 \in \mathbf{Q}(X)$ of degrees at most $2, 3$ respectively, and a monic polynomial $R(X)$ of degree 4 whose graph $Y = R(X)$ intersects the plane cubic curve $C : Y^3 + A_2(X)Y + A_3(X) = 0$ at the 12 points $P_i : (X, Y) = (x_i, R(x_i))$. Then we expect to get rank $12 - 1$ by regarding C as an elliptic curve with origin (say) P_1. Now the condition on the x_i, A_j, and R is equivalent to $\prod_{i=1}^{12}(X - x_i) = R^3 + A_2 R + A_3$. The x_i thus uniquely determine R as the principal part of the Taylor expansion at infinity of $\left(\prod_{i=1}^{12}(X - x_i)\right)^{1/3}$, and then we can recover A_2 and A_3 if and only if the X^{-1} coefficient of $\left(\prod_{i=1}^{12}(X - x_i)\right)^{1/3}$ vanishes (in which case A_2, A_3 are unique). That coefficient is a homogeneous quintic $F(x_1, \ldots, x_{12})$, which is also invariant under translation $(x_i) \mapsto (x_i - \xi)$ and thus yields degree-5 hypersurface in $\mathbf{P}^{10}$. This hypersurface contains some obvious rational subvarieties such as the subspace cut out by $x_i + x_{6+i} = 0$ ($1 \leq i \leq 6$), but this choice makes our 11 points dependent (though it gives C a 2-torsion point and can thus be used to construct elliptic curves of moderately large rank with $T \supseteq \mathbf{Z}/2\mathbf{Z}$). Mestre finds a less obvious rational subvariety of dimension 3 that preserves independence, consisting of arbitrary linear combinations of

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[4] Four new points are visible, but the two points in each double section sum to an element of the known rank-8 group.

$(a, a, a, b, b, b, c, c, c, d, d, d)$ and $(b, c, d, a, c, d, a, b, d, a, b, c)$ for fixed a, b, c, d.[5] Variations of this idea were later used by Mestre and others [see table above] to push the record rank over $\mathbf{Q}(t)$ to 14 (and also for other purposes, such as constructing hyperelliptic curves of given genus with many rational points). Moreover, it was the rank-11 family that was used by Mestre, Nagao, Fermigier, Kouya, and Martin-McMillen during 1992–2000 to raise the rank record for individual curves $E/\mathbf{Q}$ from 14 to 15, 17, 19, 20, 21, 22, 23, and finally 24 (see [Dujella 2006a]), even after curves of rank > 11 over $\mathbf{Q}(t)$ were found, because Mestre's curves have simpler coefficients and more parameters, and thus offer greater scope for searching for high-rank specializations.

What of the rank of elliptic curves $E/F(T)$ for general fields F? We exclude the case of a "constant elliptic curve", in which E is isomorphic over $F(T)$ with a fixed elliptic curve E_0/F, because then $E(F(T)) = E_0(F)$ (proof: no nonconstant maps from $\mathbf{P}^1$ to E_0), which can be very large and/or complicated if F is large enough. For nonconstant curves, the geometry of the associated elliptic surface (more on this in the next lecture) yields the result that $E(F(T))$ is finitely generated. The list of possible torsion groups is the same that Mazur proved over $\mathbf{Q}$ together with $(\mathbf{Z}/N\mathbf{Z})^2$ ($N = 3, 4, 5$) and $(\mathbf{Z}/3\mathbf{Z}) \oplus (\mathbf{Z}/6\mathbf{Z})$, and the proof is much easier than over $\mathbf{Q}$. But, as with Mordell's theorem for $E/\mathbf{Q}$, the bound on the rank is not uniform. Indeed, when F has characteristic $p > 0$ the rank is unbounded for "isotrivial" curves with constant and supersingular j-invariant [Šafarevič–Tate 1967], and also for non-isotrivial ones [Ulmer 2002]. In characteristic zero, it is not yet known whether the rank of nonconstant elliptic curves over $F(T)$ is unbounded even for $F = \mathbf{C}$; the record is due to Shioda [1992]: the curve $y^2 = x^3 + T^n + 1$ over $\mathbf{C}(T)$ has (trivial torsion and) rank ≤ 68, with equality if and only if $360|n$. Note that even though the curve is defined over $\mathbf{Q}$, most sections are not; for instance, if $3|n$ then $(x, y) = (-\mu T^{n/3}, 1)$ is on the curve for each $\mu \in \mathbf{C}$ such that $\mu^3 = 1$. Still, the generators are all defined over some number field F_0, and it follows by Néron's specialization theorem that there are infinitely many elliptic curve of rank at least 68 over F_0.

II Elliptic surfaces and K3 surfaces.

[This part began with a review of the general setup of elliptic curves E over $F(T)$ for an arbitrary field $F \subset \mathbf{C}$, relating the arithmetic of E with intersection theory on the corresponding elliptic surface $\mathcal{X}$. We do not repeat all of this material here; see for instance [Shioda 1990].]

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[5] An extra dimension can be obtained by adding multiples of $(c, d, b, d, a, c, b, d, a, c, a, b)$ and $(d, b, c, c, d, a, d, a, b, b, c, a)$, the latter of which is redundant but highlights the $\mathcal{A}_4$ symmetry. This symmetry suggests the following equivalent construction of the resulting copy of $\mathbf{P}^2 \times \mathbf{P}^2$ in the hypersurface $F_5 = 0$: let V be the irreducible 3-dimensional representation of the alternating group $\mathcal{A}_4$, let $\langle \cdot, \cdot \rangle$ be an $\mathcal{A}_4$-invariant perfect pairing on V, and let v, v' be any vectors in V; then the 12 inner products $\langle v, gv' \rangle$ ($g \in \mathcal{A}_4$) are coordinates x_i of a point on F_5. This can be verified by regarding $F_5(x_1, \ldots, x_{12})$ as an $\mathcal{A}_4^2$-invariant polynomial on $V \oplus V$ and showing it must vanish identically [NDE 3.vii.1991, unpublished e-mail to J.-F. Mestre].

We assume throught that $\mathcal{X}$ is a minimal Néron model for E. Such a surface $\mathcal{X}$ is birational with an elliptic surface in extended Weierstrass form $y^2 + a_1 xy + a_3 = x^3 + a_2 x^2 + a_4 x + a_6$, with each a_i $(i = 1, 2, 3, 4, 6)$ a section of $O(id)$ for some nonnegative integer d (in down-to-earth language, a homogeneous polynomial of degree id in two variables). The smallest such d is the "arithmetic genus" of $\mathcal{X}$. As the name suggests, the description of elliptic surfaces of arithmetic genus d gets more complicated as d increases. When $d = 0$ we have a constant elliptic curve E_0 over $F(T)$ (equivalently, a surface $\mathcal{X} \cong E_0 \times \mathbf{P}^1$). Once $d > 0$, it follows from intersection theory on $\mathcal{X}$, together with the fact that $h^{1,1}(\mathcal{X}) = 10d$, that the Mordell–Weil rank of $E/F(t)$ is at most $10d - 2$. Except for the smallest few d it is not known whether this upper bound can be attained.

When $d = 1$ we say $\mathcal{X}$ is a "rational elliptic surface", because it is birational with $\mathbf{P}^2$, at least over an algebraic closure $\overline{F}$. Néron's surfaces of rank 8 are rational. Since $8 = 10d - 2$ for $d = 1$, this gives the maximal Mordell–Weil rank of a rational elliptic surface. Much more can be said of the geometry and arithmetic of such surfaces, notably Shioda's beautiful work relating rational elliptic surfaces with the invariant theory of the Weyl group of the root lattice E_8 and its root sublattices; but we shall not follow this thread here.

Our main concern is the case $d = 2$, when $\mathcal{X}$ is an "elliptic K3 surface". A K3 surface is a smooth algebraic surface $\mathcal{X}$ with trivial canonical class and $H^1(\mathcal{X}, O_{\mathcal{X}}) = 0$. This is the last case in which an algebraic surface can be elliptic in more than one way; we heavily exploit this flexibility in our analysis and computations.

A key invariant of a K3 surface $\mathcal{X}$ is its *Néron–Severi lattice* $\mathrm{NS}(\mathcal{X}) = \mathrm{NS}_{\overline{F}}(\mathcal{X})$. The Néron–Severi lattice of any compact algebraic surface over F is its Néron–Severi group (divisors defined over $\overline{F}$ modulo algebraic equivalence), equipped with the symmetric integer-valued pairing induced from the intersection pairing on divisors. For a K3 surface, this group is a free abelian group, and the pairing is *even*: $D \cdot D \in 2\mathbf{Z}$ for all $D \in \mathrm{NS}(\mathcal{X})$. Let ρ be the rank of $\mathrm{NS}(\mathcal{X})$. By the index theorem, the pairing is nondegenerate of signature $(1, \rho - 1)$.

If $\mathcal{X}$ is elliptic then $\mathrm{NS}(\mathcal{X})$ contains two distinguished classes defined over F, the fiber f (preimage of any point under the map $T : \mathcal{X} \to \mathbf{P}^1$) and the zero-section s_0. The intersection pairing on the subgroup H they generate is determined by $f \cdot f = 0$, $s_0 \cdot f = 1$, and $s_0 \cdot s_0 = -d = -2$; hence H is isomorphic with the "hyperbolic plane" (i.e., the even unimodular lattice with Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). Conversely, *any* copy of H in $\mathrm{NS}(\mathcal{X})$ defined over F yields a model of $\mathcal{X}$ as an elliptic surface: one of the generators or its negative is effective, and has 2 independent sections, whose ratio gives the desired map to $\mathbf{P}^1$. (Warning: in general one might have to subtract some base locus to recover the fiber class f.) Moreover, the pair $(\mathrm{NS}(\mathcal{X}), H)$ determines the reducible fibers[6] and Mordell–Weil

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[6] Except for the distinctions between Kodaira types I_1 and II (simple node and cusp, neither of which contributes to $\mathrm{NS}(\mathcal{X})$), I_2 and III (either of which contributes A_1), and I_3 and IV (either of which contributes A_2).

group of the elliptic surface over $\overline{F}$. Indeed let N_{ess}, the "essential lattice" of the elliptic surface, be the orthogonal complement of H in $\mathrm{NS}(\mathcal{X})$, with the pairing scaled by -1 to make it positive definite. Let $R \subseteq N_{\text{ess}}$ be the *root lattice* of N_{ess}, the sublattice spanned by the roots (vectors of norm 2) in N_{ess}. This is a direct sum of root lattices A_n ($n \geq 1$), D_n ($n \geq 4$), or E_n ($n = 6, 7, 8$), with each factor indicating a reducible fiber of the corresponding type; and the Mordell–Weil group $E(\overline{F}(T))$ is canonically isomorphic with the quotient group N_{ess}/R. In particular its rank is the difference between the ranks of N_{ess} and R. The rank of N_{ess}, in turn, equals $\rho - 2$, so the Mordell–Weil rank is at most $\rho - 2$, with equality if and only if N_{ess} has no roots; in this case the Mordell–Weil rank over $F(T)$ is also $\rho - 2$ if and only if $\mathrm{NS}(\mathcal{X})$ consists entirely of divisor classes defined over F.

Now ρ is at most $h^{1,1}(\mathcal{X}) = 10d = 20$, whence the upper bound $18 = 20 - 2$ on the Mordell–Weil rank. While a rational surface always has $\rho = h^{1,1}$, for more complicated surfaces the Néron–Severi rank ρ may be strictly smaller. For K3 surfaces over $\mathbf{C}$ the situation is completely described by the Torelli theorem of Piateckii-Shapiro and Šafarevič [1971]. This theorem confirms and refines the following naïve parameter count: there are $9 + 13 - 4 = 18$ parameters for an elliptic K3 surface (the coefficients a_4, a_6 of a narrow Weierstrass model have $8 + 1$ and $12 + 1$ coefficients, and we subtract 4 for the dimension of GL_2 acting on the projective coordinates of the T-line); each free N_{ess} generator, whether in R or the Mordell–Weil group, imposes one condition and thus reduces the dimension of the moduli space by 1. Recall that over $\mathbf{C}$ it is known that $H^2(\mathcal{X}, \mathbf{Z}) \cong \mathrm{II}_{3,19}$ (the unique even unimodular lattice of signature $(3, 19)$), and that $\mathrm{NS}(\mathcal{X})$ embeds into $H^2(S, \mathbf{Z})$. The Torelli theorem asserts that this embedding is "optimal", that is, realizes $\mathrm{NS}(\mathcal{X})$ as the intersection of $H^2(\mathcal{X}, \mathbf{Z})$ with a $\mathbf{Q}$-vector subspace of $H^2(\mathcal{X}, \mathbf{Z}) \otimes \mathbf{Q}$; for every such lattice L of signature $(1, \rho - 1)$, there is a nonempty (coarse) moduli space of pairs $(\mathcal{X}, \iota)$, where $\iota : L \to \mathrm{NS}(\mathcal{X})$ is an optimal embedding consistent with the intersection pairing; and each component of the moduli space has dimension $20 - \rho$. Moreover, for $\rho = 20, 19, 18, 17$ these moduli spaces repeat some more familiar ones: CM elliptic curves for $\rho = 20$, elliptic and Shimura modular curves for $\rho = 19$, and moduli of abelian surfaces and RM abelian surfaces for certain cases of $\rho = 17$ and $\rho = 18$. It turns out that many of those moduli spaces are more readily parametrized via K3 surfaces than by more direct approaches. We shall treat these applications elsewhere, concentrating here on the application to elliptic K3 surfaces.

To attain the upper bound of 18 on the Mordell–Weil rank, we must use a model of one of the (countably infinite number of) K3 surfaces of Néron–Severi rank 20 as an elliptic surface with trivial R. This can happen over $\mathbf{C}$, and thus over $\overline{\mathbf{Q}}$ [Cox 1982, Nishiyama 1995]; these proofs via [Piateckii-Shapiro–Šafarevič 1971] use transcendental methods and yield no explicit equations, but the example $Y^2 = X^3 - 27(T^{12} - 11T^6 - 1)$ was later obtained in [Chahal–Meijer–Top 2000]. This still leaves open the question of whether an elliptic K3 surface can have Mordell–Weil rank 18 over $\mathbf{Q}(T)$. We repeat the warning that it is not sufficient for the surface to be defined over $\mathbf{Q}$; as with Shioda's surface $Y^2 = X^3 + T^{360} + 1$,

the Chahal–Meijer–Top surface does not have all of $\mathrm{NS}(\mathcal{X})$ defined over $\mathbf{Q}$. Likewise for the Néron–Severi groups of some other familiar examples of K3 surfaces of maximal Néron–Severi rank, such as the diagonal quartic $X^4 + Y^4 = Z^4 + T^4$ in $\mathbf{P}^3$ or the complete intersection $\sum_{i=1}^6 X_i = \sum_{i=1}^6 X_i^2 = \sum_{i=1}^6 X_i^3 = 0$ in $\mathbf{P}^5$.

One somewhat familiar example where the full Néron–Severi group *is* defined over $\mathbf{Q}$ is the universal elliptic curve with a 7-torsion point, considered naturally as an elliptic surface over the modular curve $X_1(7) \cong \mathbf{P}_\mathbf{Q}^1$. But this surface has $|\mathrm{disc}(\mathrm{NS}(\mathcal{X}))| = 7$, much too small for any of its elliptic-surface models to have rank 18. In fact we combine arithmetic considerations with the construction in [Inose 1978] to show that if $\mathrm{NS}_\mathbf{Q}(\mathcal{X})$ has rank 20 then $\mathrm{disc}(\mathrm{NS}(\mathcal{X}))$ is one of the thirteen discriminants $-3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67,$ -163 of imaginary quadratic orders of class number 1. Each of these arises uniquely up to twists, albeit with different elliptic models — already -3 and -4 have 6 and 13 respectively. But even 163 is too small for N_{ess} to have no roots.[7] Therefore there are no elliptic K3 surfaces of Mordell–Weil rank 18 over $\mathbf{Q}$.[8] But Mordell–Weil rank 17 is barely possible — still not with $\big(\rho, |\mathrm{disc}(\mathrm{NS}(\mathcal{X}))|\big) = (20, 163)$ but with an exceptional rational point on a certain Shimura curve!

More on this soon; first we describe torsion on elliptic K3 surfaces. Each of the torsion groups in Mazur's list, other than $\mathbf{Z}/9\mathbf{Z}$, $\mathbf{Z}/10\mathbf{Z}$, $\mathbf{Z}/12\mathbf{Z}$, and $(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/8\mathbf{Z})$, can arise for such a surface, requiring at least the following reducible fibers, and thus giving an upper bound on the rank equal to 6 less than the number of degenerate fibers counted *without* multiplicity:

torsion	$\{0\}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$	$\mathbf{Z}/4\mathbf{Z}$	$\mathbf{Z}/5\mathbf{Z}$
fibers	1^{24}	$2^8 1^8$	$3^6 1^6$	$4^4 2^2 1^4$	$5^4 1^4$
formula	$(0,0,0,a_4,a_6)$	$(0,a_2,0,a_4,0)$	$(a_1,0,a_3,0,0)$	$(a_1,a_2,a_1a_2,0,0)$	etc.
bound	18	10	6	4	2

torsion	$\mathbf{Z}/6\mathbf{Z}$	$\mathbf{Z}/7\mathbf{Z}$	$\mathbf{Z}/8\mathbf{Z}$	$(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/2\mathbf{Z})$	$(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/4\mathbf{Z})$	$(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/6\mathbf{Z})$
fibers	$6^2 3^2 2^2 1^2$	$7^3 1^3$	$8^2\, 4\, 2\, 1^2$	2^{12}	$4^4 2^4$	$6^3 2^3$
bound	2	0	0	6	2	0

The three cases with bound zero are the universal elliptic curves with that torsion group. When the bound is positive it can always be attained over $\mathbf{C}$ but (as was already seen in the case of trivial torsion) might not be attainable over $\mathbf{Q}$. The maximal rank is not known yet in each case, because with nontrivial torsion it is possible for the Mordell–Weil group to be defined over $\mathbf{Q}$ even though $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts nontrivially on $\mathrm{NS}_{\overline{\mathbf{Q}}}(\mathcal{X})$. Still, the discriminant -163 surface does have an elliptic model that attains rank 4 with torsion group $\mathbf{Z}/4\mathbf{Z}$, and was

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[7] Such a lattice, positive-definite of rank 18 with discriminant 163 and minimal norm ≥ 4, would have broken the density record for a sphere packing in $\mathbf{R}^{18}$. But the existence of such a lattice is not excluded by known sphere-packing bounds, so its impossibility had to be proved by other means.

[8] This was asserted in [Shioda 1994], but as a consequence of an incorrect result that was later retracted.

used to get rank 12 over $\mathbf{Q}$ (the previous rank record for an elliptic curve with a rational 4-torsion point was 9, by Kulesz–Stahlke 2001). Explicitly, the surface has equation $Y^2 + aXY + abY = X^3 + abX^2$ where $(a, b) = ((8T - 1)(32T + 7), 8(T+1)(15T-8)(31T-7))$; it has a 4-torsion point at $X = Y = 0$, and four points with $X = -15(T+1)(31T-7)(32T+7)/4$, $(8T-1)(15T-8)(31T-7)(32T+7)$, $-(T+1)(8T-1)(15T-8)(32T+7)$, and $-4(T+1)(2T+5)(15T-8)(32T+7)$ that together with torsion generate $E(\mathbf{Q}(T))$; and taking $T = 18745/6321$ yields a curve $E/\mathbf{Q}$ with eight further independent points, so $E(\mathbf{Q}) \cong (\mathbf{Z}/4\mathbf{Z}) \oplus \mathbf{Z}^{12}$. There are also various ways to combine pairs of quadratic sections to get infinitely many $E/\mathbf{Q}$ with $E(\mathbf{Q}) \supseteq (\mathbf{Z}/4\mathbf{Z}) \oplus \mathbf{Z}^6$; the previous rank record for an infinite family with 4-torsion was 5 (Kihara 2004, according to [Dujella 2006], where he cites two papers in *Proc. Japan Acad. A*).

A variant approach is to get some of the torsion group by a suitable base change; for instance our records with torsion group $(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/2\mathbf{Z})$ were obtained by starting from an elliptic K3 surface of Néron–Severi rank 20 with torsion group $\mathbf{Z}/2\mathbf{Z}$ whose remaining 2-torsion points are defined over a quadratic extension of $\mathbf{Q}(T)$ that is still rational; likewise we obtained $(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/4\mathbf{Z})$ by quadratic base change from a curve over $\mathbf{Q}(T)$ with torsion group $\mathbf{Z}/4\mathbf{Z}$.

We return now to the problem of finding elliptic K3 surfaces of large Mordell–Weil rank with no torsion restriction. Having proved that rank 18 is unattainable, we try for rank 17, corresponding to Néron–Severi rank 19. Here the moduli spaces have dimension $20 - 19 = 1$, and in principle the Torelli theorem for K3 surfaces [Piateckii-Shapiro–Šafarevič 1971] identifies these curves with standard arithmetic quotients. In practice it is not always easy to identify the modular curve corresponding to a given lattice L of signature $(1, 18)$, especially when we need results over $\mathbf{Q}$ rather than $\mathbf{C}$. But some identifications can be made. For instance, If $L \supset \mathrm{II}_{1,17}$ then the surfaces are parametrized by the classical modular curve $X_0(N)/w_N$ where $N = \mathrm{disc}(L)/2$. This curve is rational for some rather large N (largest is 71), and elliptic of rank 1 for some N that are even larger (largest is 131). For $N > 131$ the curve has only finitely many rational points by Mordell-Faltings. We need only one rational point, but it must be neither a cusp (because cusps yield degenerate surfaces) nor a CM point (because CM points yield a surfaces of rank 20). It is expected that there are only finitely many examples; the largest known are for $N = 191$ [Elkies 1998] and $N = 311$ [Galbraith 1999]. But again even those N are too small for N_{ess} to have no roots. Still, $N = 311$ is large enough for R to have rank only 2, leaving Mordell–Weil rank $17 - 2 = 15$. This was already a new record, and as with Néron's construction it could be pushed a bit further with quadratic sections, to 16 over $\mathbf{Q}(T)$ and 17 for infinitely many specializations. (We can increment only once over $\mathbf{Q}(T)$, because for elliptic K3 surfaces we cannot choose the ramification points.) But I did not compute explicit equations for this K3 surface: such a computation would have been a huge undertaking then, and even now with better tools it would be a substantial project. I did, however, manage to compute an elliptic model for the K3 surface for the $N = 191$ point that has a 2-torsion point and the minimal

root lattice A_1^8 that can accommodate $\mathbf{Z}/2\mathbf{Z}$ torsion. Thus this elliptic surface has Mordell–Weil group $(\mathbf{Z}/2\mathbf{Z}) \oplus \mathbf{Z}^9$ over $\mathbf{Q}(T)$. Quadratic sections increment this to 10 over $\mathbf{Q}(T)$ and 11 for an infinite family, improving on Kihara's 2001 and 1997 records of 9 ([Dujella 2006], again citing papers in *Proc. Japan Acad. A*). Specialization of the K3 surface to $t \in \mathbf{Q}$ produced the new record curve $E/\mathbf{Q}$ with $E(\mathbf{Q}) \cong (\mathbf{Z}/2\mathbf{Z}) \oplus \mathbf{Z}^{18}$.

One can do even better when L is an even lattice of signature $(1, 18)$ that does not contain $\mathrm{II}_{1,17}$. Let $N = \mathrm{disc}(L)/2$, and suppose N is squarefree. Then $L \supset \mathrm{II}_{1,17}$ if and only if a certain obstruction in $\mathrm{Br}_2(\mathbf{Q})$ vanishes. This obstruction is supported on an even subset of the prime factors of N. If it does not vanish then we get the corresponding Shimura modular curve instead of a classical (elliptic) modular curve. When N is composite, the Shimura curve can have smaller genus than $X_0(N)/w_N$ because there are fewer oldforms. This lets us use N large enough that N_{ess} can have trivial root system. Even so, we did not find any case where the Shimura curve has infinitely many rational points. But for $N = 6 \cdot 79$ we found a sporadic non-CM point. Here the Shimura curve has genus 2, and a bielliptic involution that lets us predict an equation for the curve using the methods of [González–Rotger 2004]. We find $u^2 = 16t^6 - 19t^4 + 88t^2 - 48$, with the following rational points: the fixed points of the bielliptic involution $(t, u) \leftrightarrow (-t, -u)$, with $t = \infty$; four points with $|t| = 2$ and $|u| = 32$; and four with $|t| = 14/13$ and $|u| = 2^6 251/13^3$. It turns out that the last orbit is non-CM. This one orbit of rational points yields an elliptic K3 surface of Mordell–Weil rank 17 over $\mathbf{Q}(t)$, answering the question in [Shioda 1994] on the maximal Mordell–Weil rank of such a surface. It also yields the new records of 18 for the Mordell–Weil rank of a nonconstant elliptic curve over $\mathbf{Q}(T)$ (again via quadratic base change), and of 19 for a lower bound on $\limsup(r)$ over curves $E/\mathbf{Q}$. Specialization of the rank-17 surface also yields several examples of elliptic curves over $\mathbf{Q}$ with more than 24 independent rational points, including a curve of rank at least 28.

Some remarks on the computation of these families and specializations are in the third lecture. We conclude this second lecture by noting that the connection between K3 surfaces of Néron–Severi rank 19 and Shimura curves also makes it possible to compute explicit information (equations, CM coordinates, Clebsch-Igusa invariants, etc.) about Shimura curves of levels considerably beyond what was previously feasible.

III Computation and results.

We briefly describe the steps of the computations needed to get from the above theory of K3 surfaces and their moduli to explicit elliptic curves over $\mathbf{Q}(T)$ and $\mathbf{Q}$.

Finding suitable positive-definite lattices N_{ess}. After the second lecture's forced march through K3 territory, I thought better of attempting another such review of Euclidean and hyperbolic lattices. Basically N_{ess} is obtained as a suitable slice of a Niemeier lattice. The Niemeier lattices are the 24 even unimodular lattices Λ of rank 24, each with a known root system R and "glue group" Λ/R, which gives

a handle on the torsion and roots of its slices. See [Conway–Sloane 1993, Ch. 10.3, pages 399–402] for an example of this technique.

Finding equations for $E/\mathbf{Q}(T)$ and its Mordell–Weil generators. Here it may seem that we are back where we started: we still seek the coefficients of polynomial identities, such as $y_i(T)^2 = x_i(T)^3 + a_4(T)x_i(T) + a_6(T)$ ($1 \leq i \leq 17$), with various auxiliary conditions on the (x_i, y_i) to ensure the correct height pairings. There are too many variables to solve such a nonlinear system directly, but in the 4-torsion case shown earlier it was barely possible. Still it was more convenient to eliminate only some of the variables, and recover the remaining ones as follows. The general theory tells us that the coefficients are rational and behave well modulo suitable small primes p such as $41 = (163 + 1)/4$. An exhaustive search mod p finds a solution. Lift this solution arbitrarily to characteristic zero and regard the lift as a p-adic approximation to the correct solution. Apply the natural generalization of Newton's iteration $x \mapsto x - F(x)/F'(x)$ to this context, using finite differences rather than derivatives to approximate F'. Each step doubles the p-adic precision. Soon the p-adic approximation is close enough to recognize the actual rational numbers by lattice reduction. Then confirm them by substitution into the desired identities. Finally change coordinates to simplify the equations to ones whose coefficients have smaller heights, which is essential for finding high-rank specializations.

Exploiting different elliptic models of the same surface. Simple example: the Inose surfaces $Y^2 = X^3 + AT^4X + B''T^7 + BT^6 + b'T^5$ over the T-line have essential lattice $N_{\text{ess}} = R = E_8^2$ with reducible fibers at $T = 0$ and $T = \infty$. Scaling to $Y^2 = X^3 + AX + B''T + B + B'/T$ we obtain an elliptic model over the X-line, this time with $R = D_{16}$ and $[N_{\text{ess}} : R] = 2$ (note that $(T, Y) = (0, 0)$ is a 2-torsion point). It turns out that the transformation is particularly simple when, as here, the two lattices are "2-neighbors": they have isomorphic index-2 sublattices. We start from a model of $\mathcal{X}$ as an elliptic surface whose coefficients are easier to compute, and then follow a chain of 2-neighbors (and the occasional 3-neighbor) that introduces or removes roots and torsion until it reaches an elliptic surface with the desired essential lattice.

Parametrizing families of K3 surfaces of Néron–Severi rank 19. When the Néron–Severi rank is 19 rather than 20, our task is not to solve for the coefficients of a single surface but to parametrize a one-dimensional family by a modular curve. We start at a known point P_0 of the curve (maybe coming from a surface of rank 20, in an elliptic model in which R is the same but the Mordell–Weil rank is larger by 1), and then deform it p-adically. For example, fix a rational function f of the coefficients (say, the cross-ratio of the T-coordinates of four reducible fibers), and use Newton to find points for which f is near $f(P_0)$. Now the coefficients are generally no longer rational even if $f(P)$ is, but they are algebraic with degree at most $\deg(f)$. We can guess those with lattice reduction if $\deg(f)$ is small enough. Varying $f(P)$ over simple rational numbers in a p-adic neighborhood of $f(P_0)$, we can then guess the equations relating those coefficients with f by solving simultaneous linear equations. At this point we have a guess for a (usually very singular)

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model for the moduli curve, and if we can't or won't find a smooth model directly then we can ask Magma (or someday Sage) for it. We then verify that the equations we guessed numerically actually work symbolically. Then specialize to the non-CM point to find the desired surface.

Incrementing the rank via quadratic base change. As already noted, the necessary "quadratic sections" — rational curves on $\mathcal{X}$ that intersect the fiber f with multiplicity 2 — are harder to find than in Néron's situation. The trace of the quadratic section is an element of $E(\mathbf{Q}(T))$, defined mod $2E(\mathbf{Q}(T))$ when we translate by elements of the Mordell–Weil group; equivalently, a half-lattice vector mod $E(\mathbf{Q}(T))$. Intersection theory tells us that we need a coset of $2E(\mathbf{Q}(T))$ in $E(\mathbf{Q}(T))$ consisting of vectors of norm 2 mod 4, with no representatives of norm less than 10. (This is for a surface with no reducible fibers; when $R \neq \{0\}$ the criterion is more complicated.) The corresponding coset of $E(\mathbf{Q}(T))$ in $\frac{1}{2}E(\mathbf{Q}(T))$ then consists of "holes" of norm at least $5/2$. For our rank-17 surface, there are literally thousands of such holes, and for each one we get an inequivalent quadratic section. The resulting genus-zero curves are all rational because we can always find some other divisor whose intersection with the quadratic section is odd. This also gives millions of biquadratic base changes of genus 1 and positive rank, any one of which gets the lower bound 19 on $\limsup(r)$. (Alas none of them degenerates to a genus-zero curve, so we do not find an elliptic curve of Mordell–Weil rank at least 19 over $\mathbf{Q}(T)$ this way.)

Guessing good specializations by Mestre's heuristic. The conjecture of Birch and Swinnerton-Dyer suggests that large rank should correlate with small partial products of $L(E, 1)$. Taking logarithms, we want to make $\sum_{p<x} \log(p/N_p)$ very negative. Experimentally, we need literally thousands of primes, and must canvass many millions of specializations E_t. That's a lot of N_p's to compute. But each depends only on t mod p, so we precompute $\sum_{p<x} p$ of them once and for all, store a low-precision approximation to $\log(p/N_p)$ for each one, and then search for large values of $\sum_p \log(N_p/p)$ in sieve style.

Finding extra rational points. The resulting candidates for large rank have coefficients much too large for it to be feasible to find new rational points by direct search. The simplest independent set of 28 rational points we could find on our record curve

$$y^2 + xy + y = x^3 - x^2 - 20067762415575526585033208209338542750930230312178956502\ x$$
$$+ 34481611795030556467032985690390720374855944359319180361266008296291939448732243429$$

has 28-digit integers for its x-coordinates! When the curve has nontrivial 2-torsion, Cremona's program MWRANK quickly computes Selmer 2-groups to find upper bounds on the rank, and then usually finds enough generators on the candidate record curves. But in the absence of torsion the coefficients are much too large for descent to be feasible. (This is why we can only say that the curve has rank at least 28, not exactly 28, though it seems quite unlikely that the rank is even larger.) Instead we exploit the known rank-17 sublattice of $E/\mathbf{Q}$ to search for rational points near half-lattice holes of the sublattice. This yields equations $y^2 = Q(x)$

for quartics Q with much smaller coefficients. (This looks close enough to the behavior of 2-descents that the method might be regarded as a fake 2-descent.) We then use a sieve technique, implemented by C. Stahlke and M. Stoll in their C program RATPOINTS, to find a few such rational points near some of the deepest half-lattice holes in the generic Mordell–Weil lattice. Finally we use the canonical height to determine the rank of the subgroup of $E(\mathbf{Q})$ generated by all the known points.

Summary of new rank records. In the following table of record ranks of families of elliptic curves with specified torsion group, "$r+$" means rank at least r over $\mathbf{Q}(T)$, and at least $r+1$ for an infinite family parametrized by a positive-rank elliptic curve obtained by quadratic base change from the record curve over $\mathbf{Q}(T)$. Plain r is a lower bound on the rank of a curve over $\mathbf{Q}(T)$. The previous records as of 2004 are from [Dujella 2006].

torsion	$\{0\}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$	$\mathbf{Z}/4\mathbf{Z}$	$(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/2\mathbf{Z})$	$(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/4\mathbf{Z})$
≤ 2004	14	9	6	5	6	3
new	18+	10+	7	5+	7+	3+

For other torsion groups, the records remain 3 for $\mathbf{Z}/5\mathbf{Z}$ and $\mathbf{Z}/6\mathbf{Z}$; "1+" for $\mathbf{Z}/7\mathbf{Z}$, $\mathbf{Z}/8\mathbf{Z}$, and $(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/6\mathbf{Z})$ (the three cases where the universal elliptic curve is K3); and "0+" for $\mathbf{Z}/9\mathbf{Z}$, $\mathbf{Z}/10\mathbf{Z}$, $\mathbf{Z}/12\mathbf{Z}$, and $(\mathbf{Z}/2\mathbf{Z}) \oplus \mathbf{Z}/8\mathbf{Z}$ (the four cases where the universal elliptic curve has $d > 2$).

The new rank records for individual curves over $\mathbf{Q}$ are as follows:

torsion	$\{0\}$	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/4\mathbf{Z}$	$\mathbf{Z}/8\mathbf{Z}$	$(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/2\mathbf{Z})$	$(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/4\mathbf{Z})$	$(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/6\mathbf{Z})$
≤ 2004	24	15	9	5	10	6	5
new	28	18	12	6	14	8	6

The incremental improvements for torsion groups $\mathbf{Z}/8\mathbf{Z}$ and $(\mathbf{Z}/2\mathbf{Z}) \oplus (\mathbf{Z}/6\mathbf{Z})$ are due only to better searching in known families. The absence (so far?) of a new record for $\mathbf{Z}/3\mathbf{Z}$ may be due to the lack of an efficient implementation of descent via a 3-isogeny.

We conclude with a few remarks on integral points. Our $r \geq 28$ curve has at least 1174 pairs $(x, \pm y)$ of integral points in its minimal model, but this is not a record: a curve with $r \geq 25$ in the same family has at least 2810 such pairs in the known subgroup of $E(\mathbf{Q})$. The same family also contains a curve for which we found only 21 independent points but the subgroup they generate contains at least 2564 pairs of integral points. Over $\mathbf{Q}(T)$, the analogue of integral points is points (X, Y) where X, Y are polynomials of degree at most $4, 6$ respectively. In the absence of reducible fibers, these are exactly the nonzero elements of the Mordell–Weil group whose canonical height is as small as 4. Our elliptic K3 surface of rank 17 has 1311 such pairs $(X, \pm Y)$.

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